

AD-A116 323

WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER  
FEEDBACK STABILIZATION OF 'HYBRID' BILINEAR SYSTEMS.(U)

F/G 12/1

APR 82 M SLEMROD, E L ROGERS

DAAG29-80-C-0041

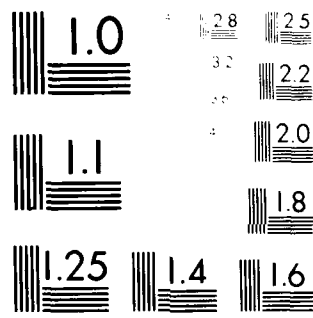
UNCLASSIFIED

MRC-TSR-2360

NL

1 01 1  
411.5  
16.528

END  
DATE  
FILMED  
107-82  
DTIC



MICROCOPY RESOLUTION TEST CHART  
NATIONAL BUREAU OF STANDARDS-1963-A

AD A116323

MRC Technical Summary Report #2360

FEEDBACK STABILIZATION OF "HYBRID"  
BILINEAR SYSTEMS

M. Slemrod and E. L. Rogers

Mathematics Research Center  
University of Wisconsin-Madison  
610 Walnut Street  
Madison, Wisconsin 53706

April 1982

Received March 18, 1982

Approved for public release  
Distribution unlimited

Sponsored by  
U. S. Army Research Office  
P. O. Box 12211  
Research Triangle Park  
North Carolina 27709

Air Force Office of  
Scientific Research  
Washington, D. C. 20332

DTIC  
S JUN 29 1982  
A

82 06 29 032

DTIC FILE COPY

UNIVERSITY OF WISCONSIN - MADISON  
MATHEMATICS RESEARCH CENTER

FEEDBACK STABILIZATION OF "HYBRID" BILINEAR SYSTEMS

M. Slemrod<sup>†</sup> and E. L. Rogers<sup>\*</sup>

Technical Summary Report #2360  
April 1982

ABSTRACT

This paper considers the problem of stabilizing a control system governed by a combination of partial and ordinary differential equations. The partial differential equations govern the evolution of the system in the interior of some spatial domain, the ordinary differential equations describe the evolution of the boundary data; the control enters through the boundary ordinary differential equations in a bilinear fashion. We provide sufficient conditions for feedback stabilization of such "hybrid" systems. Two examples to wave equations with dynamic boundary conditions are provided.

AMS(MOS) Subject Classifications: 93B05, 93C10, 93C20, 93C25

Key Words: feedback control, distributed parameter systems,  
bilinear control system.

Work Unit Number 1 (Applied Analysis)

---

†

Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, New York 12181 and the Mathematics Research Center, University of Wisconsin, Madison, Wisconsin 53706.

\*

Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, New York 12181.

---

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and in part by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Contract/Grant No. AFOSR-81-0172. The United States Government is authorized to reproduce and distribute reprints for Government purposes not withstanding any copyright hereon.

## SIGNIFICANCE AND EXPLANATION

In the control of distributed parameter systems i.e., systems governed by partial differential equations, controls often times can be applied only on the boundary of some spatial domain. In this paper we consider the case when the controls enter the boundary conditions as bilinear (in the control and state) ordinary differential equations. We show how we can synthesize feedback controls that will stabilize systems whose uncontrolled motion is critically stable.



The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

# FEEDBACK STABILIZATION OF "HYBRID" BILINEAR SYSTEMS

M. Slemrod<sup>†</sup> and E. L. Rogers<sup>\*</sup>

## 0. Introduction

In a recent paper [1] Burns and Cliff formulated a model of a "hybrid" system, i.e., a mixed system of partial and ordinary differential equations, in which the control enters only within the context of the ordinary differential equations. Interest in such systems can be motivated by problems in structures in which the control dynamics take place only in the boundary conditions of a distributed parameter system. In this paper we consider two problems (motivated by the example of Burns and Cliff) where the control enters the boundary conditions in a bilinear fashion. Our goal is to synthesize feedback controls which will stabilize the originally critically stable systems, i.e., we wish to find a feedback controls so that the states of the feedback systems approach the zero state as  $t \rightarrow \infty$ .

The main tool of our analysis will be "hyperbolic" stabilization theory of Ball and Slemrod [2], [3]. In fact the theory of [2], [3] may be readily applied to our problem.

The paper is divided into five parts. Section 1 recalls the results on feedback stabilization of [2], [3]. Section 2 introduces the first hybrid system. Section 3 shows the feedback stabilizability of the first hybrid system. Similarly Section 4 discusses the second hybrid system and Section 5 proves feedback stabilizability for it.

---

†

Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, New York 12181 and the Mathematics Research Center, University of Wisconsin, Madison, Wisconsin 53706.

\*

Department of Mathematical Sciences, Rensselaer Polytechnic Institute, Troy, New York 12181.

---

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and in part by the Air Force Office of Scientific Research, Air Force Systems Command, USAF, under Contract/Grant No. AFOSR-81-0172. The United States Government is authorized to reproduce and distribute reprints for Government purposes not withstanding any copyright hereon.

# 1. Feedback Stabilization.

Consider the initial value problem

$$\dot{u}(t) = Au(t) + p(t) u(t), \quad (P)$$

$$u(0) = u_0 \in X,$$

where  $A$  is the infinitesimal generator of a  $C_0$  semigroup of contractions  $e^{At}$  on a real Hilbert space  $X$ ,  $B$  is a bounded linear operator  $X \rightarrow X$ , and  $p(t)$  is a real valued control.  $X$  is endowed with inner product  $\langle \cdot, \cdot \rangle_X$ .

Definition System  $(P)$  is stabilized (weakly stabilizable) if there exists a continuous feedback control  $p: X \rightarrow \mathbb{R}$  such that  $(P)$  with  $p(t) = p(u(t))$  satisfies the following properties.

- (i) For each  $u_0$  there exists a unique weak solution of  $u(t; u_0)$  defined for all  $t \in \mathbb{R}$ , of  $(P)$ .
- (ii)  $\{0\}$  is a stable equilibrium of  $(P)$ .
- (iii)  $u(t, u_0) \rightarrow 0$  ( $u(t, u_0) \rightarrow 0$  weakly) as  $t \rightarrow \infty$  in  $X$  for all  $u_0 \in X$ .

The natural approach to the stabilization problem is to differentiate  $\|u(t)\|_X^2 = \langle u(t), u(t) \rangle_X$  along trajectories of  $(P)$ . In this manner we obtain

$$\frac{d}{dt} \|u(t)\|_X^2 = 2 \langle Au(t), u(t) \rangle_X + 2p(t) \langle u(t), Bu(t) \rangle_X,$$

at least formally. An obvious choice of feedback control (though not the only one) is

$$p(u) = -\langle u, Bu \rangle_X,$$

since this control yields the "dissipative energy inequality"

$$\frac{d}{dt} \|u(t)\|_X^2 \leq -2 \langle u(t), Bu(t) \rangle_X^2.$$

(Note that  $e^{At}$  a contraction means  $\langle Au(t), u(t) \rangle_x \leq 0$ , for  $u(t) \in D(A)$ ).

So formally this choice of control  $p(u)$  yields a feedback system of that form

$$\dot{u}(t) = Au(t) - \langle u(t), Bu(t) \rangle_x u(t). \quad (F)$$

For the purposes of this paper the following theorem of Ball and Slemrod [3] will be needed.

Theorem 1. If  $B: X \rightarrow X$  is compact and

$$\langle e^{At}\psi, Be^{At}\psi \rangle_x = 0 \text{ for all } t \in \mathbb{R}^+ \implies \psi = 0 \quad (C)$$

then (P) is weakly stabilizable.

In some applications it will be convenient to work with second order "hyperbolic" systems of the form

$$\ddot{y}(t) + Ay(t) + p(t)By(t) = 0, \quad (P')$$

$$y(0) = y_0 \in H_A, \quad \dot{y}(0) = y_1 \in H.$$

Here  $H$  is a real Hilbert space with inner product  $\langle \cdot, \cdot \rangle_H$  and norm  $\| \cdot \|_H^2 = \langle \cdot, \cdot \rangle_H$ .  $A$  is a densely defined positive self-adjoint operator on  $H$  such that  $A^{-1}$  is everywhere defined and compact. We suppose the eigenvalues  $\lambda_n^2$  of  $A$ ,  $n = 1, 2, \dots$ ,  $0 < \lambda_1 < \lambda_2 < \dots$ , are simple. We denote the corresponding sequence of eigenfunctions by  $\{\phi_n\}$ . Let  $H_A = D(A^{1/2})$ .

$H_A$  form a Hilbert space under the inner product

$$\langle y, y^* \rangle_A = \langle A^{1/2}y, A^{1/2}y^* \rangle.$$

Denote  $\| \cdot \|_A$  as the norm of  $H_A$ . We assume  $B$  is a bounded linear map:  $H_A \rightarrow H$ .

Again the function  $p(t)$  is a scalar real valued control.

If we set

$$u(t) = \begin{pmatrix} y(t) \\ \dot{y}(t) \end{pmatrix}, \quad A = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 \\ -B & 0 \end{pmatrix}, \quad D(A) = D(A) \times H_A,$$

$$X = H_A \times H \quad \text{with} \quad \langle (y, \dot{y}), (y^*, \dot{y}^*) \rangle_X = \langle y, y^* \rangle_{H_A} + \langle \dot{y}, \dot{y}^* \rangle$$

we see  $A, B$  are required for problem (P) above. We can now state the theorem of Ball and Slemrod given above in a second order context (see [2] for a generalization).

Theorem 2. Suppose  $A$  and  $B$  are as above. Assume  $B : H_A \rightarrow H$  is compact and

$$(H1) \quad \langle B\phi_k, \phi_k \rangle_H \neq 0 \quad \text{for } k = 1, 2, \dots$$

$$(H2) \quad \lambda_m \pm \lambda_n \neq 2\lambda_k \quad \text{unless } m = n = k \text{ and the } + \text{ sign is taken,}$$

or both  $\langle B\phi_m, \phi_n \rangle_H$  and  $\langle B\phi_n, \phi_m \rangle_H$  are zero.

Then the feedback system

$$\ddot{y}(t) + Ay(t) + \langle By(t), \dot{y}(t) \rangle_H By(t) = 0 \quad (F')$$

with

$$p(y(t), \dot{y}(t)) = \langle By(t), \dot{y}(t) \rangle_H$$

and the initial data  $y(0) = y_0 \in H_A$ ,  $\dot{y}(0) = y_1 \in H$  possesses a unique globally defined weak solution  $(y, \dot{y}) \in C((0, \infty); H_A \times H)$  and  $(y(t), \dot{y}(t)) \rightarrow (0, 0)$  weakly in  $X = H_A \times H$  as  $t \rightarrow \infty$  and  $(P')$  is weakly stabilizable.

Proof.  $B : H_A \rightarrow H$  compact implies  $B : X \rightarrow X$  is compact and (H1), (H2) imply (C). See [2] for details.

## 2. Hybrid system 1.

Motivated by the example in [1] we consider the string-mass system shown in Figure 1. The string has length 1, constant linear density  $\sigma$ , and is under constant tension  $\tau$ . The purpose of the device at the right end is to maintain the tension, however it is idealized so as to provide no impedance to the vertical motion at the end.

The vertical motion of the string is assumed to satisfy the linear wave equation

$$z_{tt}(t,x) = \alpha^2 z_{xx}(t,x), \quad 0 < x < 1, \quad (2.1)$$

where  $\alpha^2 = \tau/\sigma > 0$ . The motion of the right hand end of the string is governed by the balance of linear momentum. The relevant forces here are acceleration of the point mass  $m$ , the tensile form on the string  $\tau z_x(t,1)$ , and external forces. We assume we can impose an external force at the right end in the bilinear form  $p(t)z(t,1)$ . In the absence of other external forces the equation of balance of linear momentum of the mass  $m$  is

$$mz_{tt}(t,1) = -p(t)z(t,1) - \tau z_x(t,1). \quad (2.2)$$

The left hand is assumed fixed at  $x = 0$  so that

$$z(t,0) = 0. \quad (2.3)$$

The initial conditions are

$$z(0,x) = f(x), \quad z_t(0,x) = g(x), \quad 0 < x < 1. \quad (2.4)$$

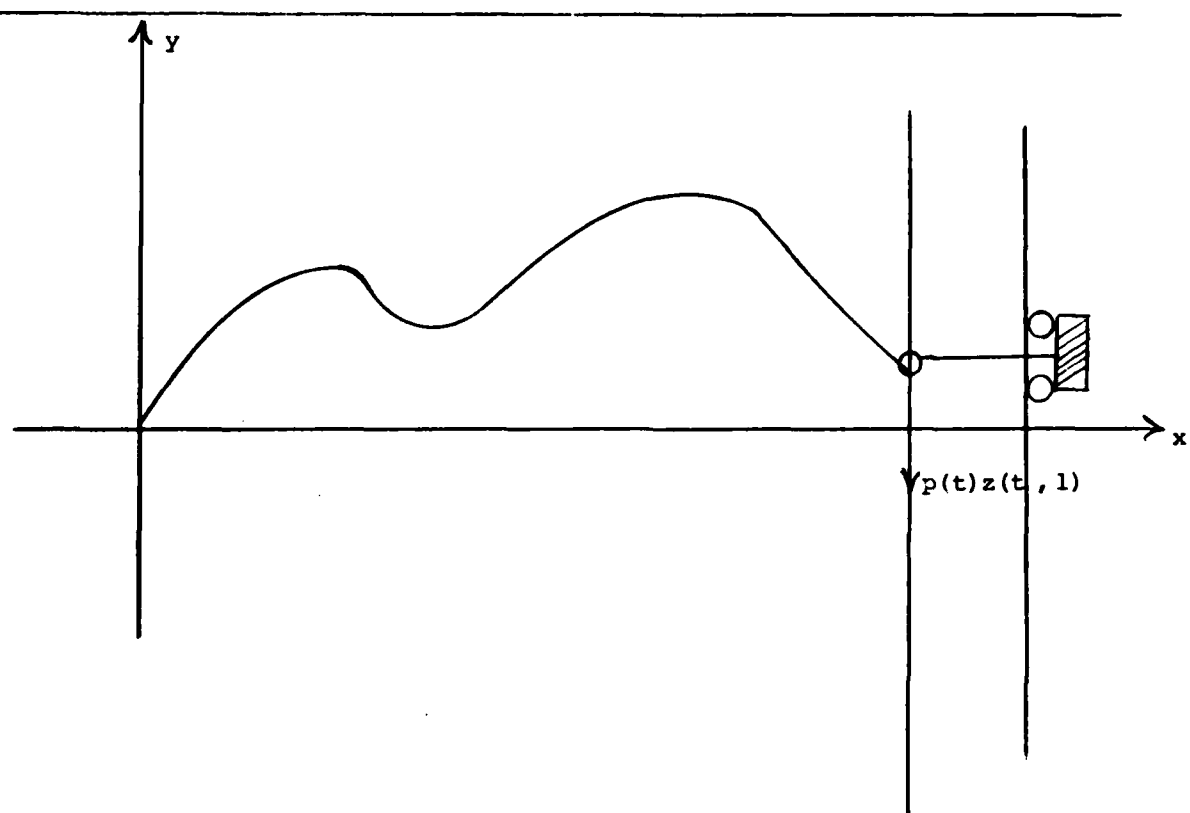


Figure 1

## 2. Hybrid system 1.

Motivated by the example in [1] we consider the string-mass system shown in Figure 1. The string has length 1, constant linear density  $\sigma$ , and is under constant tension  $\tau$ . The purpose of the device at the right end is to maintain the tension, however it is idealized so as to provide no impedance to the vertical motion at the end.

The vertical motion of the string is assumed to satisfy the linear wave equation

$$z_{tt}(t,x) = \alpha^2 z_{xx}(t,x), \quad 0 < x < 1, \quad (2.1)$$

where  $\alpha^2 = \tau/\sigma > 0$ . The motion of the right hand end of the string is governed by the balance of linear momentum. The relevant forces here are acceleration of the point mass  $m$ , the tensile form on the string  $\tau z_x(t,1)$ , and external forces. We assume we can impose an external force at the right end in the bilinear form  $p(t)z(t,1)$ . In the absence of other external forces the equation of balance of linear momentum of the mass  $m$  is

$$mz_{tt}(t,1) = -p(t)z(t,1) - \tau z_x(t,1). \quad (2.2)$$

The left hand is assumed fixed at  $x = 0$  so that

$$z(t,0) = 0. \quad (2.3)$$

The initial conditions are

$$z(0,x) = f(x), \quad z_t(0,x) = g(x), \quad 0 < x < 1. \quad (2.4)$$

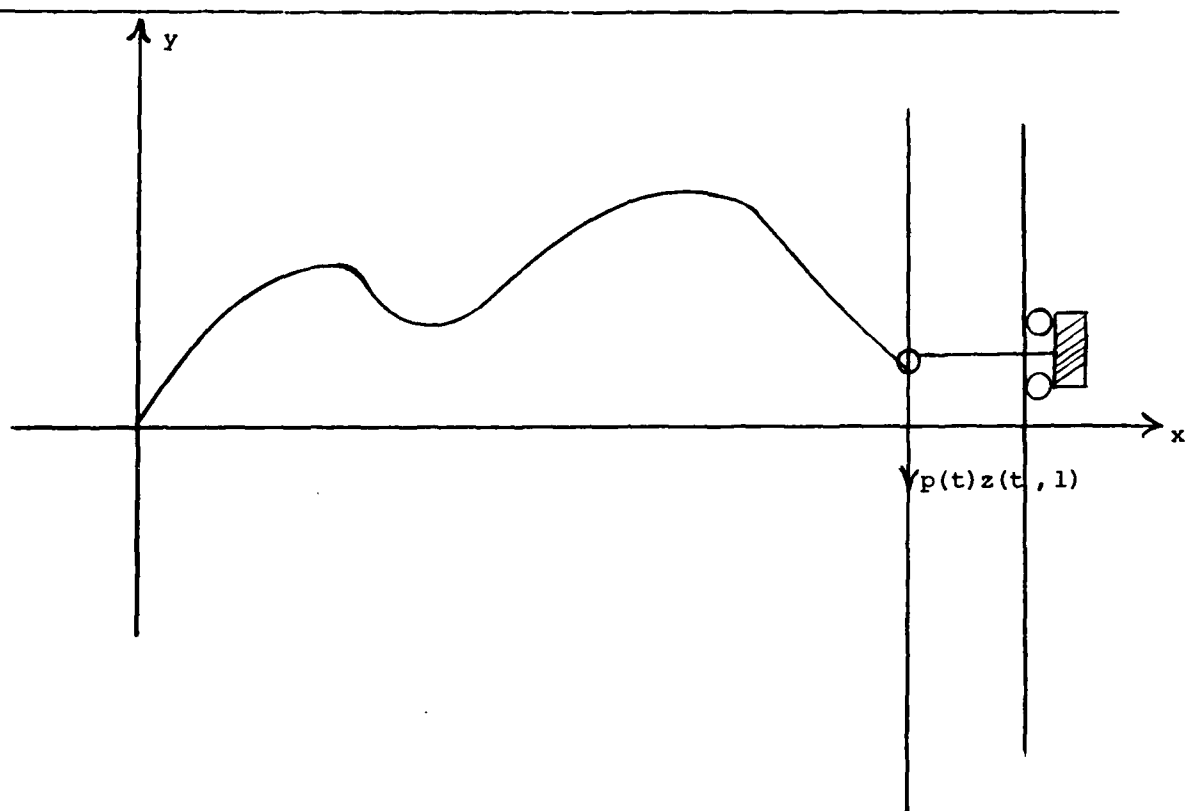


Figure 1

### 3. Abstract formulation of the hybrid system 1 and feedback stabilization.

For the hybrid system (2.1)-(2.4) set

$$y(t) = \begin{pmatrix} z(t, x) \\ z_1(t) \end{pmatrix} \quad \text{where } z_1(t) = z(t, 1).$$

Also define the operators

$$A = \begin{pmatrix} -\alpha^2 \frac{d^2}{dx^2} & 0 \\ \frac{1}{m} \frac{d}{dx} \Big|_{x=1} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{m} \end{pmatrix},$$

where  $D(A) = \{(z, z_1) \in H^2(0, 1) \times \mathbb{R}; z(0) = 0, z(1) = z_1\}$ . Set

$H = L^2(0, 1) \times \mathbb{R}$  where  $H$  is a Hilbert space endowed with the inner product

$$\langle (v, w), (v^*, w^*) \rangle_H = \frac{1}{\alpha^2} \int_0^1 v(x) v^*(x) dx + \frac{m}{1} w w^*.$$

Clearly  $D(A)$  is dense in  $H$ . Of course we see that now (2.1)-(2.4) has the form (1.1).

Lemma 1.  $A$  is a positive definite, self-adjoint operator on  $H$ , with bounded inverse  $A^{-1}$ .

Proof. We first prove  $A$  is symmetric, i.e.,  $\langle A(v, w), (v^*, w^*) \rangle_H =$

$\langle (v, w), A(v^*, w^*) \rangle_H$  for all  $(v, w), (v^*, w^*) \in D(A)$  where  $D(A) \subset D(A^*)$ . To

see this simply compute

$$\begin{aligned} \langle A(v, w), (v^*, w^*) \rangle_H &= \int_0^1 v'(x) v'^*(x) dx \\ &= \langle (v, w), A(v^*, w^*) \rangle_H. \end{aligned} \quad (3.1)$$

Furthermore since for  $(v, w) \in D(A)$  we have  $v(x) = \int_0^x v'(x) dx \leq \left( \int_0^1 v'(x)^2 dx \right)^{1/2}$

and hence the inequality

$$\sup_{0 \leq x \leq 1} |v(x)|^2 + \int_0^1 v(x)^2 dx \leq \text{const.} \int_0^1 v'(x)^2 dx. \quad (3.2)$$

Thus from (3.1) we have

$$\langle A(v,w), (v,w) \rangle_H \geq \text{const.} \| (v,w) \|_H^2, \text{const.} > 0, \quad (3.3)$$

i.e., the operator  $A$  is accretive. By the Schwarz inequality we have

$$\| A(v,w) \|_H \| (v,w) \|_H \geq \text{const.} \| (v,w) \|_H^2$$

and hence

$$\| A(v,w) \|_H \geq \text{const.} \| (v,w) \|_H. \quad (3.4)$$

Thus  $A$  possesses a bounded inverse on  $H$ . Hence  $R(A) = H$  and a well known result (e.g. see Yosida [4; p. 199]) shows  $D(A^*) \subset D(A)$  and  $A$  is self adjoint. Inequality (3.3) shows  $A$  is positive definite  $\square$

Lemma 2.  $A^{-1} : H \rightarrow H$  is compact.

Proof. Let  $\{(v_n, w_n)\}$  be a bounded sequence in  $H$  and set  $A^{-1}(v_n, w_n) = (z_n, z_{1n})$ ,  $n = 1, 2, \dots$ . Since  $A^{-1}$  is bounded  $\|(z_n, z_{1n})\|_H \leq \text{const.}$  for all  $n$ . Also since

$$-\alpha^2 \frac{d^2}{dx^2} z_n(x) = v_n(x), \quad \frac{\tau}{m} \frac{dz_n(1)}{dx} = w_n. \quad (3.5)$$

Multiplication of the first equation in (3.5) by  $z_n$  and integration by parts yields

$$\begin{aligned} \alpha^2 \int_0^1 z_n'(x)^2 dx &= \alpha^2 z_n(1) z_n'(1) + \int_0^1 z_n(x) v_n(x) dx \\ &= \alpha^2 \frac{m}{\tau} z_n(1) w_n + \int_0^1 z_n(x) v_n(x) dx \\ &= \alpha^2 \langle (z_n, z_{1n}), (v_n, w_n) \rangle \\ &\leq \alpha^2 \| (z_n, z_{1n}) \|_H \| (v_n, w_n) \|_H \leq \text{const.} \end{aligned} \quad (3.6)$$

Since  $\int_0^1 z_n^2(x) dx > \int_0^2 z^2(x) dx$  by (3.2) we know  $\{z_n\}$  lies in a bounded set of  $H^1(0,1)$ . Since the injection of  $H^1(0,1)$  into  $L^2(0,1)$  is compact  $\{z_n\}$  possesses a convergent subsequence in  $L^1(0,1)$ . Since  $\{z_n\}$  is a bounded sequence in  $\mathbb{R}$  it certainly contains a bounded subsequence. Hence  $A^{-1}$  maps bounded sets of  $H$  into precompact sets of  $H$  and hence is compact.  $\square$

Lemma 3. The eigenvalues of  $A$  are given by a sequence  $0 < \lambda_1^2 < \lambda_2^2 < \dots$  where  $\{\lambda_n\}$  are the increasing positive roots of

$$\frac{\tau}{m\alpha\lambda} = \tan\left(\frac{\lambda}{\alpha}\right); \quad \{(\sin\lambda_n x, \sin\lambda_n)\}$$

are the associated eigenvectors.

Proof. If  $\lambda^2$  is an eigenvalue of  $A$ , then

$$-\alpha^2 \frac{d^2 z}{dx^2} = \lambda^2 z, \quad 0 < x < 1, \quad \frac{\tau}{m} \frac{dz(1)}{dx} = \lambda^2 z(1),$$

and  $z(0) = 0$ , where  $(z(x), z(1))$  is the associated eigenvector. Thus  $z(x) = \sin\left(\frac{\lambda}{\alpha}x\right)$  and

$$\frac{\tau}{m\alpha\lambda} = \tan\left(\frac{\lambda}{\alpha}\right). \quad \square$$

Lemma 4.  $B$  is a bounded linear self-adjoint operator:  $H \rightarrow H$ .

Proof.  $\|B(v, w)\|_H = \|(0, \frac{w}{m})\|_H \leq \text{const.} \|(v, w)\|_H$  so  $B$  is bounded. Also  $\langle B(v, w), (v^*, w^*) \rangle_H = \frac{1}{\tau} w w^* = \langle (v, w), B(v^*, w^*) \rangle_H$  so  $B$  is symmetric and bounded, hence self-adjoint.  $\square$

Lemma 5.  $\langle B(\sin\lambda_n x, \sin\lambda_n), (\sin\lambda_n x, \sin\lambda_n) \rangle_H \neq 0$  for  $n = 1, 2, \dots$ .

Proof.  $\langle B(\sin\lambda_n x, \sin\lambda_n), (\sin\lambda_n x, \sin\lambda_n) \rangle_H = \frac{\sin^2}{\tau} \lambda_n \neq 0$ .

Lemma 6. For the eigenvalues of  $A$  we know  $\lambda_m \pm \lambda_n \neq 2\lambda_k$  unless  $m=n=k$  and the  $+$  sign is taken.

Proof. Assume  $\lambda_m \pm \lambda_n = 2\lambda_k$ . Divide by  $\alpha$  and take  $\tan$  of both sides.

We then have

$$\tan\left(\frac{\lambda_m}{\alpha} \pm \frac{\lambda_n}{\alpha}\right) = \tan\left(\frac{2\lambda_k}{\alpha}\right)$$

which in turn implies by the usual trigonometric identities

$$\frac{\tan\left(\frac{\lambda_m}{\alpha}\right) \pm \tan\left(\frac{\lambda_n}{\alpha}\right)}{1 \mp \tan\left(\frac{\lambda_m}{\alpha}\right) \tan\left(\frac{\lambda_n}{\alpha}\right)} = \frac{2 \tan \frac{\lambda_k}{\alpha}}{1 - \tan^2\left(\frac{\lambda_k}{\alpha}\right)}.$$

We now use the definition of  $\lambda_n$  (Lemma 3) to assert

$$\frac{\frac{\tau}{m\alpha} (\lambda_m^{-1} \pm \lambda_n^{-1})}{1 \mp \left(\frac{\tau}{m\alpha}\right)^2 \lambda_m^{-1} \lambda_n^{-1}} = \frac{2\left(\frac{\tau}{m\alpha}\right) \lambda_k^{-1}}{1 - \left(\frac{\tau}{m\alpha}\right)^2 \lambda_k^{-2}}$$

or

$$\frac{(\lambda_n \pm \lambda_m)}{\lambda_m \lambda_n \mp \left(\frac{\tau}{m\alpha}\right)^2} = \frac{2\lambda_k}{\lambda_k^2 - \left(\frac{\tau}{m\alpha}\right)^2}. \quad (3.7)$$

We now have two cases to consider.

Case 1.  $\lambda_m + \lambda_n = 2\lambda_k$ .

In this case (3.7) implies  $\lambda_m \lambda_n = \lambda_k^2$ . But since  $\lambda_m^2 + \lambda_n^2 + 2\lambda_m \lambda_n = 4\lambda_k^2$  we see  $(\lambda_m - \lambda_n)^2 = 0$  and  $\lambda_m = \lambda_n = \lambda_k$ .

Case 2.  $\lambda_m - \lambda_n = 2\lambda_k$ .

In this  $\lambda_n - \lambda_m = -2\lambda_k$  and (3.4) implies

$$\lambda_m \lambda_n + \left(\frac{\tau}{m\alpha}\right)^2 = -\lambda_k^2 + \left(\frac{\tau}{m\alpha}\right)^2$$

or

$\lambda_m \lambda_n = -\lambda_k^2$  which contradicts the positivity of  $\{\lambda_n\}$ .

This completes the proof.  $\square$

Theorem 3. The feedback system (2.1), (2.2) with

$$p(t) = \frac{z_t(t,1)z(t,1)}{m}$$

possesses a unique globally defined weak solution for initial data  $(f, f(1)) \in$

$H_A$ ,  $(g, g(1)) \in H$ ,  $((y, y_1), (\dot{y}, \dot{y}_1)) \rightarrow ((0,0), (0,0))$  weakly in  $H_A \times H$  as

$t \rightarrow \infty$ , and (2.1), (2.2) is weakly stabilizable in  $H_A \times H$ .

Proof. Lemmas 1-6 show the hypotheses of Theorem 2 are satisfied.  $\square$

#### 4. The hybrid system 2.

Suppose  $G$  is a domain in  $\mathbb{R}^3$  filled with a compressible fluid which is at rest except for acoustic wave motion. If  $\phi(x,y,z,t)$  is the velocity potential, so that  $-\nabla\phi$  is the particle velocity, then linearized theory says that  $\phi$  satisfies the wave equation

$$\phi_{tt} = c^2 \Delta \phi \quad \text{in } G, \quad (4.1)$$

where  $c$  is the speed of sound in the medium,  $\phi_t$  is the pressure distribution ( $= w(x,y,z,t)$ ). Now suppose that the boundary of  $G$   $\partial G$  possesses a non-rigid section  $\Gamma_1$  which is subject to small oscillations. We assume that each point on  $\Gamma_1$  reacts to the excess pressure of the acoustic wave like a harmonic oscillator. We assume also that different parts of the boundary do not influence each other, so that  $\Gamma_1$  is locally reacting. Then the normal displacement  $\delta$  of the boundary  $\Gamma_1$  into  $G$  satisfies an equation of the form

$$m \delta_{tt} + k\delta = -\rho\phi_t + f \quad \text{on } \Gamma_1, \quad (4.2)$$

where  $\rho$  is the (assumed constant in linear theory) density of the fluid,  $m, k$  are positive constants, and  $f$  is an applied external force on  $\Gamma_1$ . If we assume the boundary section  $\Gamma_1$  is impenetrable we obtain a third equation from the continuity of velocity at the boundary,

$$\delta_t = \frac{\partial \phi}{\partial n} \quad \text{on } \Gamma_1, \quad (4.3)$$

where  $\frac{\partial \phi}{\partial n}$  is the outward normal velocity.

(This model has been given in [5], p. 263; a mathematical analysis is found in

(This model has been given in [5], p. 263; a mathematical analysis is found in [6], and a discussion of linear control synthesis appears in [7].)

If we assume  $G$  is bounded and  $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 = \partial G$  where  $\Gamma_2$  is a free surface then the pressure on  $\Gamma_2$  must take on the ambient atmospheric pressure distribution. If we normalize this boundary pressure distribution to zero we find

$$\phi_t = 0 \quad \text{on } \Gamma_2. \quad (4.4)$$

$\Gamma_3$  is a rigid boundary so

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{on } \Gamma_3. \quad (4.5)$$

In this paper we consider the case where  $G$  is a right circular cylinder with free surface at the top and reacting boundary at the bottom. This is shown in Figure 2.

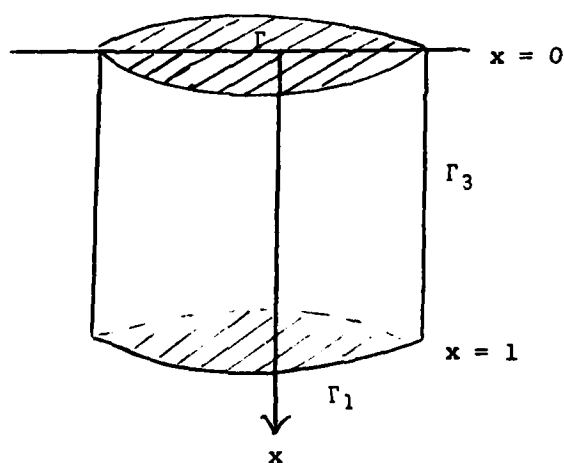


Figure 2

For simplicity we set  $c = \rho = m = k = 1$ . Also we seek a solution of (4.1) - (4.5)  $\phi = \phi(x, t)$ . In this case (4.5) is automatically satisfied.

Differentiation of (4.1) - (4.4) with respect to  $t$  yields the system

$$w_{tt} = w_{xx} \quad 0 < x < 1 \quad (4.1')$$

$$\theta_{tt} + \theta = -w_t(t, 1) + v(t)w(t, 1), \quad x = 1, \quad (4.2')$$

$$\theta_t = w_x, \quad x = 1, \quad (4.3')$$

$$w = 0, \quad x = 0, \quad (4.4')$$

where  $\theta \equiv \delta_t$  and we have synthesized the external driving force  $f'(t) = v(t)w(t, 1)$ , i.e., we are looking for a feedback which then multiplies the observed pressure at  $x = 1$ . Of course we must also specify initial conditions

$$w(0, x) = w_0(x), \quad w_t(0, x) = w_1(x), \quad \theta(0) = \theta_0, \quad \theta_t(0) = \theta_1, \quad 0 < x < 1.$$

# 5. Abstract formulation of hybrid system 2 and feedback stabilization.

For the hybrid system (4.1') - (4.4') set

$$u = \begin{pmatrix} w \\ w_t \\ \theta \\ \theta_t \end{pmatrix}, \quad Au = \begin{pmatrix} w_t \\ w_{xx} \\ \theta_t \\ -\theta - w_t(\cdot, 1) \end{pmatrix}, \quad Bu = \begin{pmatrix} 0 \\ 0 \\ 0 \\ w(\cdot, 1) \end{pmatrix},$$

$X = \{(w, w_t, \theta, \theta_t); w \in H^1(0, 1), w_t \in L_2(0, 1), \theta \in \mathbb{R}, \theta_t \in \mathbb{R}, w = 0 \text{ at } x = 0\}$

endowed with inner product

$$\langle u, u^* \rangle_X = \int_0^1 (w_x w_x^* + w_t w_t^*) dx + \theta \theta^* + \theta_t \theta_t^*,$$

$$D(A) = \{(w, w_t, \theta, \theta_t) \in X; w \in H^2(0, 1), w_t \in H^1(0, 1),$$

$$w_t = 0 \text{ at } x = 0, \theta_t = w_x \text{ at } x = 1\}.$$

Since for  $w \in H^1(0, 1)$  the Sobolev lemma ([4], p. 174]) says  $w$  can be viewed as an element of  $C(0, 1)$  after possible modification on a set measure zero, we shall take such  $w$  as a continuous. So for  $w \in H^1(0, 1)$ ,  $w = 0$ ,  $w$  is identified as a continuous function which satisfies

$$w(x) = \int_0^x w'(x) dx$$

in the Lebesgue sense. Hence the boundary value  $w(1)$  is simply

$$w(1) = \int_0^1 w'(x) dx. \quad (5.1)$$

It is in this sense that the boundary conditions in  $X$  and  $D(A)$  are to be understood.

We note for  $u \in D(A)$ ,  $\langle Au, u \rangle_X = 0$  so  $A$  is certainly dissipative.  $A$  is also densely defined and  $\text{Range}(I - A) = X$  (by a direct computation) so the Lumer-Phillips Theorem [4] implies,  $A$  is the infinitesimal generator of a  $C_0$  semigroup of contractions  $e^{At}$  on  $X$ . Also we note that if  $\{u_n\}$  is a bounded sequence in  $X$ , (5.1) shows  $\{w_n(1)\}$  to be a bounded sequence in  $\mathbb{R}$ . Hence it possesses a convergent subsequence. Thus  $B : X \rightarrow X$  is compact.

All the hypotheses of Theorem 1 are satisfied modulo showing that the only solution of  $\langle e^{At}\psi, e^{At}\psi \rangle = 0$  for all  $t \in \mathbb{R}^+$  is  $\psi \equiv 0$ . In this problem

$$\langle e^{At}\psi, e^{At}\psi \rangle = \tilde{\theta}_t \tilde{w}(1, t)$$

where  $\tilde{w}, \tilde{\theta}$  are solutions to the uncontrolled ( $p \equiv 0$ ) system (4.1') -

(4.4'). Separation of variables shows

$$\tilde{w}(x, t) = \sum_{k=1}^{\infty} (A_k e^{i\sigma_k t} + \bar{A}_k e^{-i\sigma_k t}) \sin \sigma_k x, \quad (5.2)$$

$$\tilde{\theta}_t(t) = \sum_{k=1}^{\infty} (A_k e^{i\sigma_k t} + \bar{A}_k e^{-i\sigma_k t}) \sigma_k \cos \sigma_k \quad (5.3)$$

where  $\sigma_k$  are the positive roots of

$$\tan \sigma = \frac{1}{\sigma} - \sigma. \quad (5.4)$$

Here  $\bar{\phantom{x}}$  denotes complex conjugate. If we substitute (5.2), (5.3) into the

equation  $\tilde{\theta}_t \tilde{w}(1, t) = 0$  for all  $t \in \mathbb{R}^+$ , we obtain

$$\begin{aligned} & 2 \sum_{k=1}^{\infty} |A_k|^2 \sigma_k \sin \sigma_k \cos \sigma_k \\ & + \sum_{k=1}^{\infty} (A_k^2 e^{i2\sigma_k t} + \bar{A}_k^2 e^{-i2\sigma_k t}) \sigma_k \sin \sigma_k \cos \sigma_k \\ & + \sum_{\substack{n, k=1 \\ n \neq k}}^{\infty} (A_n \bar{A}_k e^{i(\sigma_n + \sigma_k)t} + \bar{A}_n \bar{A}_k e^{i(\sigma_n - \sigma_k)t} \\ & + A_n \bar{A}_k e^{i(\sigma_n - \sigma_k)t} + \bar{A}_n \bar{A}_k e^{i(\sigma_k - \sigma_n)t}) \sigma_n \sin \sigma_k \cos \sigma_n \\ & = 0 \quad \text{for all } t \in \mathbb{R}^+. \quad (5.5) \end{aligned}$$

The terms on the left hand side of (5.5) form an almost periodic function in

$t$ . If none of the frequencies appearing in the second sum appear in the first

and third sums the uniqueness theorem for almost periodic functions will imply

$A_k = 0, k = 1, 2, \dots$ . This in turn will imply  $\tilde{w}(x, t) = 0, \tilde{\theta}_t(t) = 0$  which

with (4.2') shows  $\psi \equiv 0$  will be the only solution of  $\langle e^{At}\psi, e^{At}\psi \rangle = 0$  for

all  $t \in \mathbb{R}^+$ . So we now prove the following lemma.

Lemma 7. Let  $\sigma_n$  be the positive roots of (5.4),  $0 < \sigma_1 < \sigma_2 < \sigma_3 \dots$ .

Then  $\sigma_r \pm \sigma_s = 2\sigma_k$  holds only when  $r=s=k$  and  $+$  sign is taken.

Proof. For later use we record the first three positive roots of (5.4):

$$\sigma_1 = 0.67625\dots$$

$$\sigma_2 = 2.11708\dots$$

$$\sigma_3 = 4.92125\dots$$

Case 1: Assume  $\sigma_r + \sigma_s = 2\sigma_k$ . Then taking tan of both sides and using (5.4)

we find

$$\frac{1 - \sigma_r \sigma_s}{4\sigma_k^2 - 3\sigma_r \sigma_s - (1 - \sigma_r \sigma_s)^2} = \frac{1 - \sigma_k^2}{\sigma_k^2 - (1 - \sigma_k^2)^2}.$$

Solving for  $\sigma_r \sigma_s$  we see

$$\sigma_r \sigma_s = \frac{3\sigma_k^2 - 2}{1 - \sigma_k^2}. \quad (5.6)$$

(Note  $\sigma_r \sigma_s \neq 1$ ,  $\sigma_k^2 \neq 1$  so our manipulations are valid). If  $\sigma_k^2 < 2/3$  or

$\sigma_k^2 > 1$  we see  $\sigma_r \sigma_s < 0$  which violates positivity of the roots. Since

$\sigma_k^2 \notin (2/3, 1)$  for all  $k$ , we see that  $\sigma_r + \sigma_s = 2\sigma_k$  holds only when

$\sigma_r = \sigma_s = \sigma_k$ , i.e.,  $r=s=k$ .

Case 2. Assume  $\sigma_r - \sigma_s = 2\sigma_k$  and proceed as in Case 1. We find this time

that

$$\sigma_k^2 = 1 + \frac{1}{\sigma_r \sigma_s - 3}. \quad (5.7)$$

If  $r, s \geq 2$ ,  $r \neq s$ , graphical analysis of (5.4) shows  $\sigma_r - \sigma_s > \sigma_3 - \sigma_2$

$= 2.80417\dots$ . So if  $2\sigma_k = \sigma_r - \sigma_s$  we must have  $\sigma_k^2 > 1.965\dots$ . Also

we know for  $r, s \geq 2$   $\sigma_r \sigma_s > 3$ . Hence from (5.7) we see that if

$\sigma_r - \sigma_s = 2\sigma_k$  for  $r, s \geq 2$ ,  $r \neq s$ , that

$$1.965\ldots < 1 + \frac{1}{\sigma_r \sigma_s - 3}$$

for  $3 < \sigma_r \sigma_s < 4.035\ldots$ . No pair  $\sigma_r, \sigma_s$  satisfy this relation.

On the other hand for  $r$  or  $s < 2$  we can only have  $s = 1, r > 1$ . We consider the set  $\{\sigma_r\}$  so that

$$1 + \frac{1}{\sigma_1 \sigma_r - 3} < \sigma_2^2 = 4.48203\ldots \quad (5.8)$$

This set is the set of  $\{\sigma_r\}$  so that  $\sigma_1 \sigma_r > 3.287186\ldots$  or  $\sigma_r > 4.8609\ldots$ , i. e.,  $r > 3$ . So for  $r > 3$  the only way (5.7) can hold with  $s = 1$  is when  $k = 1$ , i. e.,  $\sigma_r - \sigma_1 = 2\sigma_1$  or  $\sigma_r = 3\sigma_1$ . But there is no such  $\sigma_r$ . Finally if  $r = 2$  we must have  $\sigma_2 - \sigma_1 = 2\sigma_k$ , i. e.,  $\sigma_k = .720415$  which also cannot hold. The lemma is proven. []

We can now conclude from the discussion preceeding Lemma 7 that the following theorem is true.

Theorem 4. The feedback system (4.1') - (4.4') with

$$v(t) = -\theta_t(t)w(1,t)$$

possesses a unique globally defined weak solution for initial data

$(w_0, w_1, \theta_0, \theta_1) \in X$  and  $(w, w_t, \theta, \theta_t) \rightarrow (0, 0, 0, 0)$  weakly in  $X$  as  $t \rightarrow \infty$  and (4.1')-(4.4') is weakly stabilizable.

REFERENCES

1. J. A. Burns and E. M. Cliff, On control and identification of hybrid systems, AIAA Symposium on Dynamics and Control of Large Space Structures, Blacksburg, Virginia, June, 1981.
2. J. M. Ball and M. Slemrod, Nonharmonic Fourier series and the stabilization of distributed semi-linear control systems, Comm. Pure and Applied Mathematics, Vol. XXXII, 555-587 (1979).
3. J. M. Ball and M. Slemrod, Feedback stabilization of distributed semilinear controls, Applied Mathematics and Optimization, 5, 169-179 (1979).
4. K. Yosida, Functional Analysis, Springer-Verlag: New York (1971)
5. P. Morse and K. U. Ingard, Theoretical Acoustics, McGraw-Hill: New York (1968).
6. J. T. Beale, Spectral properties of an acoustic boundary condition, Indiana Univ. Math. J., 25, 895-917, (1975).
- 7., R. E. O'Brien, Energy decay in weakly locally reacting boundary value problems, to appear Indiana Univ. Math. J.

MS:ELR/db

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2360	2. GOVT ACCESSION NO. AD-A116323	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle)  FEEDBACK STABILIZATION OF "HYBRID" BILINEAR SYSTEMS		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s)  M. Slemrod and E. L. Rogers		8. CONTRACT OR GRANT NUMBER(s)  DAAG29-80-C-0041 AFOSR-81-0172
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS  1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS  See Item 18		12. REPORT DATE April 1982
		13. NUMBER OF PAGES 19
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report)  UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report)  Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES U. S. Army Research Office P. O. Box 12211 Research Triangle Park North Carolina 27709  Air Force Office of Scientific Research Washington, D. C. 20332		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)  feedback control, distributed parameter systems, bilinear control system.		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)  This paper considers the problem of stabilizing a control system governed by a combination of partial and ordinary differential equations. The partial differential equations govern the evolution of the system in the interior of some spatial domain, the ordinary differential equations describe the evolution of the boundary data; the control enters through the boundary ordinary differential equations in a bilinear fashion. We provide sufficient conditions for feedback stabilization of such "hybrid" systems. Two examples to wave equations with dynamic boundary conditions are provided.		

